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Proofs of Central-Difference Interpolation Formulas

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Using umbral calculus results we give some elegant proofs for the classical central-difference polynomial interpolation formulas.

DELTA OPERATOR AND TAYLOR EXPANSION

Let D and E denote the differentiation and forward-shift operators, respectively. A linear operator on the space of polynomials is called shift-invariant if it commutes with the operator E. Each shift-invariant operator can be expressed as a power series in D, $\sum_{n=0}^{\infty} c_n D^n$. A delta operator is a shift-invariant operator with $c_0 = 0$ and $c_1 \neq 0$. For each delta operator Q there is a unique sequence of polynomials $p_0(x)$, $p_1(x)$, $p_2(x)$,..., where $p_n(x)$ is a polynomial of degree n, $p_0(x) \equiv 1$, and $0 = p_1(0) = p_2(0) = \cdots$, such that $Qp_n(x) = np_{n-1}(x)$, $n = 1, 2, 3,...$. These polynomials have been called poweroids [10, p. 335], basic polynomials [8, p. 592; 3, p. 181] and associated polynomials [7, p. 105; 5, sect. 5].

The principal tool for our proofs is the following generalization of the Taylor expansion formula.

THEOREM [6, Sect. 5; 7, Theorem 10.c]. *Let Q be a delta operator with associated sequence of polynomials $\{p_n(x)\}$. Let T be an invertible shift-invariant operator, then*

$$E^x = \sum (T^{-1} p_n(x)/n!)\, TQ^n.$$

CENTRAL-DIFFERENCE INTERPOLATION FORMULAS

Consider the delta operator $\delta = E^{1/2} - E^{-1/2}$, called the central-difference operator. Its associated sequence of polynomials is $\{x^{[n]} = x(x + \frac{1}{2}n - 1)$

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$(x+\frac{1}{2}n-2)\cdots(x-\frac{1}{2}n+1)\}$ [9, p. 8; 4, Sect. 6.5; 7, p. 135]. The expansion formula

$$E^x = \sum \frac{x^{[n]}}{n!}\delta^n$$

is not a useful interpolation formula since it requires values at –0.5, 0.5, 1.5, etc., in addition to values at integral points.

The averaging operator $\mu = \frac{1}{2}(E^{1/2}+E^{-1/2})$ is an invertible operator, and $\mu^{-1}x^{[n]} = x^{[n+1]}/x$ [9, p. 10]. This equality is an immediate consequence of the Rodrigues formula [10, p. 338; 4, Theorem 4(4)] $p_{n+1}(x) = x\,(dQ/dD)^{-1}\,p_n(x)$, since $d\delta/dD = \mu$. Writing $x^{[n]}/x$ as $x^{[n]-1}$, we have

$$E^x = \sum \frac{x^{[n+1]-1}}{n!}\mu\delta^n.$$

This formula was derived by J. F. Steffensen in [10, p. 362].

Since $x^{[2n]} = \prod_{j=0}^{n-1}(x^2-j^2)$ and $x^{[2n+1]} = x\prod_{j=0}^{n-1}(x^2-(j+\frac{1}{2})^2)$, both $x^{[2n]}$ and $x^{[2n+1]-1}$ are even functions and both $x^{[2n-1]}$ and $x^{[2n]-1}$ are odd functions. Consequently we have

$$\tfrac{1}{2}(E^x+E^{-x}) = \sum \frac{x^{[2n]}}{2n!}\delta^{2n} = \sum \frac{x^{[2n+1]-1}}{2n!}\mu\delta^{2n},$$

and

$$\tfrac{1}{2}(E^x-E^{-x}) = \sum \frac{x^{[2n-1]}}{(2n-1)!}\delta^{2n-1} = \sum \frac{x^{[2n]-1}}{(2n-1)!}\mu\delta^{2n-1}.$$

By combining the above equations we obtain the Stirling and Bessel interpolation formulas:

$$E^x = \sum_{n=0}^{\infty} \frac{x^{[2n]}}{2n!}\delta^{2n} + \frac{x^{[2n+2]-1}}{(2n+1)!}\mu\delta^{2n+1},$$

$$E^x = \sum_{n=0}^{\infty} \frac{x^{[2n+1]-1}}{2n!}\mu\delta^{2n} + \frac{x^{[2n+1]}}{(2n+1)!}\delta^{2n+1}.$$

To derive the Everett formula, we note that

$$E^x = \frac{E-E^{-1}}{E-E^{-1}}E^x = \frac{E^x-E^{-x}}{E-E^{-1}}E + \frac{E^{1-x}-E^{-(1-x)}}{E-E^{-1}},$$

and

$$\frac{E^x - E^{-x}}{E - E^{-1}} = \frac{E^x - E^{-x}}{2\mu\delta} = \sum_{n=1}^{\infty} \frac{x^{[2n]-1}}{(2n-1)!}\,\delta^{2n-2}$$
$$= \sum_{m=0}^{\infty} \binom{x+m}{2m+1}\,\delta^{2m}.$$

To derive the Steffensen formula, consider

$$E^x = \frac{(E^{x+1/2} + E^{-(x+1/2)})\,E^{1/2} - (E^{x-1/2} + E^{-(x-1/2)})\,E^{-1/2}}{E - E^{-1}}.$$

Using the formula $\frac{1}{2}(E^y + E^{-y}) = \mu + \sum_{n=1}^{\infty} (y^{[2n+1]-1}/2n!)\,\mu\delta^{2n}$, we get

$$E^x = I + \sum_{n=1}^{\infty} \left(\binom{n+x}{2n} E^{1/2} - \binom{n-x}{2n} E^{-1/2}\right) \delta^{2n-1}.$$

To obtain the Gauss forward and backward formulas we employ the following odd–even decompositions:

$$E^x = \frac{E^x - E^{-x}}{E^{1/2} + E^{-1/2}}\,E^{1/2} + \frac{E^{x-1/2} + E^{-(x-1/2)}}{E^{1/2} + E^{-1/2}},$$

and

$$E^x = \frac{E^x - E^{-x}}{E^{1/2} + E^{-1/2}}\,E^{-1/2} + \frac{E^{x+1/2} + E^{-(x+1/2)}}{E^{1/2} + E^{-1/2}}.$$

Remarks

The traditional way of proving these formulas is to first derive the Gauss forward and backward formulas. The Stirling and Bessel formulas are obtained by averaging the two Gauss formulas, while the Everett and Steffensen formulas are obtained by eliminating from the Gauss forward formula the odd and even differences, respectively. For various symbolic calculus proofs of these formulas, we refer the reader to [9, Sect. 18; 2, p. 175, No. 5; 1, Sect. 8.5].

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